

Bareiss elimination: exactness and correctness

Luna-Flow/linear-algebra · immut determinant

This note proves the two facts that `@immut.Matrix::determinant` relies on for $n \geq 5$: every intermediate value of the fraction-free recurrence is a minor of the input, so all divisions are exact in an integral domain, and the last value is the determinant. Indices are zero-based, as in the code.

1. Setting

Let R be a commutative ring and $A = (a_{ij}) \in R^{n \times n}$. For $0 \leq k \leq n$ let M_k be the leading $k \times k$ block of A (rows and columns $0, \dots, k-1$), with $\det M_0 = 1$. For $i, j \geq k$ define the **bordered minor**

$$d_{ij}^{(k)} = \det \begin{pmatrix} M_k & c_j^{(k)} \\ r_i^{(k)} & a_{ij} \end{pmatrix},$$

where $c_j^{(k)} = (a_{0j}, \dots, a_{k-1,j})^T$ and $r_i^{(k)} = (a_{i0}, \dots, a_{i,k-1})$. In particular $d_{ij}^{(0)} = a_{ij}$, $d_{kk}^{(k)} = \det M_{k+1}$, and $d_{n-1,n-1}^{(n-1)} = \det A$.

The algorithm computes $b_{ij}^{(0)} = a_{ij}$ and, for $k = 0, \dots, n-2$ and $i, j > k$,

$$b_{ij}^{(k+1)} = \frac{b_{kk}^{(k)} b_{ij}^{(k)} - b_{ik}^{(k)} b_{kj}^{(k)}}{b_{k-1,k-1}^{(k-1)}},$$

with the convention $b_{-1,-1}^{(-1)} = 1$.

2. Sylvester's identity

Theorem. For all k and all $i, j > k$,

$$d_{ij}^{(k+1)} \cdot \det M_k = d_{kk}^{(k)} d_{ij}^{(k)} - d_{ik}^{(k)} d_{kj}^{(k)}.$$

Proof. Both sides are polynomials with integer coefficients in the entries of A . It therefore suffices to prove the identity over the field of rational functions $\mathbb{Q}(a_{00}, \dots, a_{n-1,n-1})$ in independent indeterminates; it then holds in every commutative ring by specialization. Over that field every M_k is invertible.

For invertible M_k , block elimination gives

$$\det \begin{pmatrix} M_k & c \\ r & x \end{pmatrix} = \det M_k \cdot (x - r M_k^{-1} c).$$

Write $s_{ij}^{(k)} = a_{ij} - r_i^{(k)} M_k^{-1} c_j^{(k)}$, the entries of the Schur complement A/M_k . Then $d_{ij}^{(k)} = \det M_k \cdot s_{ij}^{(k)}$.

The quotient property of Schur complements states that eliminating one more row and column is one step of Gaussian elimination on the Schur complement:

$$s_{ij}^{(k+1)} = s_{ij}^{(k)} - \frac{s_{ik}^{(k)} s_{kj}^{(k)}}{s_{kk}^{(k)}},$$

and $\det M_{k+1} = \det M_k \cdot s_{kk}^{(k)}$. Substituting,

$$\begin{aligned} d_{ij}^{(k+1)} &= \det M_{k+1} \cdot s_{ij}^{(k+1)} \\ &= \det M_k \left(s_{kk}^{(k)} s_{ij}^{(k)} - s_{ik}^{(k)} s_{kj}^{(k)} \right) \\ &= \frac{d_{kk}^{(k)} d_{ij}^{(k)} - d_{ik}^{(k)} d_{kj}^{(k)}}{\det M_k}, \end{aligned}$$

which is the claim after multiplying by $\det M_k$. □

3. Consequences

Corollary 1 (the recurrence computes minors). $b_{ij}^{(k)} = d_{ij}^{(k)}$ whenever the divisions are defined.

By induction on k : the case $k = 0$ is the definition, and the step is the theorem, because $b_{k-1,k-1}^{(k-1)} = d_{k-1,k-1}^{(k-1)} = \det M_k$.

Corollary 2 (exact division). If R is an integral domain and $\det M_k \neq 0$, the numerator of the recurrence is divisible by $\det M_k$ in R , and the quotient is the minor $d_{ij}^{(k+1)} \in R$. In an integral domain the quotient by a non-zero element is unique, so integer division in `Int` or `BigInt` returns exactly this value (absent overflow).

Corollary 3 (result). $b_{n-1,n-1}^{(n-1)} = d_{n-1,n-1}^{(n-1)} = \det A$.

4. Pivoting

Row i of $b^{(k)}$ depends only on row i of A and on rows $0, \dots, k-1$. Exchanging two rows with indices $\geq k$ before step k therefore yields exactly the values of the algorithm run on A with those rows exchanged, whose determinant is $-\det A$. The code multiplies the result by -1 for every exchange.

If every candidate pivot $b_{ik}^{(k)}$, $i \geq k$, is zero, then all bordered minors $d_{ik}^{(k)}$ vanish. Since $\det M_k \neq 0$ at this point (it is the previous pivot), this means $s_{ik}^{(k)} = 0$ for all $i \geq k$: the k -th column of the Schur complement is zero, so the Schur complement is singular and

$$\det A = \det M_k \cdot \det(A/M_k) = 0,$$

which is what the code returns.

5. Size bound

Every $b_{ij}^{(k)}$ is a $(k+1) \times (k+1)$ minor of A , so Hadamard's inequality gives $|b_{ij}^{(k)}| \leq \prod_r \|\text{row}_r\|_2$ over the rows involved, at most $H = \prod_r \max(1, \|\text{row}_r\|_2)$. The numerator of the recurrence is a difference of two products of such minors, hence bounded by $2H^2$. For fixed-width integers, no intermediate value overflows when $2H^2$ is below the largest representable integer.